MATH 124B: HOMEWORK 2

Suggested due date: August 29th, 2016

(1) Show that there is no solution of

$$\begin{cases} \Delta u = f & \text{in } D, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial D \end{cases}$$

in three dimensions, unless

$$\iiint f dV = \iint_{\partial D} g dS.$$

(2) Is the function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ harmonic? What about $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$?

(3) Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, and $0 < y < b$ with the following boundary conditions: $u_x = -a$ on $x = 0$, $u_x = 0$ on $x = a$, $u_y = b$ on $y = 0$, and $u_y = 0$ on $y = b$.

(4) Find the harmonic function on the unit square $[0,1] \times [0,1]$ with the boundary conditions $u(x,0) = x$, $u(x,1) = 0$, $u_x(0,y) = 0$, and $u_x(1,y) = y^2$.

(5) Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$.

(6) Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$ and the condition that u is bounded.

(7) Prove the uniqueness of the Robin problem

$$\begin{cases} \Delta u = f & \text{in } D \\ \frac{\partial u}{\partial n} + au = h & \text{on } \partial D, \end{cases}$$

where D is any domain in three dimensions and where a is a positive constant.

(8) Derive the three-dimensional maximum principle from the mean value property.

(9) Prove the uniqueness up to constants of the Neumann problem using the energy method.

(10) Prove Dirichlet's principle for the Neumann boundary condition in 3 dimensions. It states that among all real valued function $w(x)$ on D, the quantity

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 dV - \iint_{\partial D} hw dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$\begin{cases} -\Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} = h(x) & \text{on } \partial D. \end{cases}$$

We require that the average of the given function $h(x)$ is zero. See ex. 7.1.5 for hints.

Solutions

(1) Integrate both sides of $\Delta u = f$, then use divergence theorem.

(2) $f = \frac{1}{r}$ in polar coordinates. For 2 dimensions, the radial Laplace equation is given by

$$u_{rr} + \frac{1}{r} u_r = 0$$

and for 3 dimensions,

$$u_{rr} + \frac{2}{r} u_r = 0.$$

Since

$$f_r = -\frac{1}{r^2}$$
$$f_{rr} = \frac{2}{r^3},$$

we see that f is harmonic in 3 dimensions but not in 2.

(3) Define $\tilde{u} := u - \frac{(x-a)^2}{2} + \frac{(y-b)^2}{2}$. Then $\tilde{u}$ has the boundary conditions $\tilde{u}_x = 0$ on $x = 0$ and $x = a$ and $\tilde{u}_y = 0$ on $y = 0$ and $y = b$. The solution to the Laplace equation with vanishing Neumann condition is a constant, hence

$$u = \frac{(x-a)^2}{2} - \frac{(y-b)^2}{2} + c$$

is a solution, where c is any constant.

(4) We consider two subproblems, where we look for harmonic functions v and w that satisfy the boundary conditions

$$\begin{cases} v(x,0) = x \\ v(x,1) = v_x(0,y) = v_x(1,y) = 0 \end{cases}, \quad \begin{cases} w_x(1,y) = y^2 \\ w(x,0) = w(x,1) = w_x(0,y) = 0. \end{cases}$$

Then $u = v + w$ satisfies the original boundary conditions. Each one can be solved by separation of variables technique so that

$$v(x,y) = \frac{(1-y)}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\sinh(n\pi(y-1)),$$

where

$$A_n = -\frac{2}{\sinh(n\pi)} \int_0^1 x\cos(n\pi x)dx.$$

and

$$w(x,y) = \sum_{n=1}^{\infty} A_n \cosh(n\pi x)\sin(n\pi y),$$

where

$$A_n = \frac{2}{n\pi\sinh(n\pi x)} \int_0^1 y^2 \sin(n\pi y)dy$$

(5) Computing the Fourier coefficients, we have

$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi}(1+3\sin\phi)\sin(n\phi)d\phi$$
$$= \begin{cases} 2 & \text{if } n=0 \\ \frac{3}{a} & \text{if } n=1 \\ 0 & \text{if } n>1. \end{cases}$$

Hence,

$$u = 1 + 3\frac{r}{a}\sin\theta.$$

(6) The solution can be obtain by an inversion of r and a, hence

$$u = 1 + \frac{3a}{r}\sin\theta$$

(7) Let v be another solution, then their difference $w := u - v$ satisfies

$$\begin{cases} \Delta w = 0 & \text{in } D \\ \frac{\partial w}{\partial n} + aw = 0 & \text{on } \partial D. \end{cases}$$

By Green's first identity, we have

$$\iint_{\partial D} w\frac{\partial w}{\partial n}dS = \iiint_D |\nabla w|^2 dV$$

Inserting the equation on the boundary, we have

$$0 \geq -a\iint_{\partial D} w^2 dS = \iiint_D |\nabla w|^2 dV \geq 0$$

hence $w = 0$.

(8) The maximum principle is the following

Proposition. If D is any solid region, a nonconstant harmonic function in D cannot take its maximum value inside D, but only on ∂D.

We prove this assuming the mean value property. Let $x_0 \in D$ be a maximum point of u, say $u(x_0) = M$. Choosing polar coordinates centered at x_0, consider a circle of radius $\varepsilon > 0$ small enough so that it lies in D. Then

$$u(x_0) = \frac{1}{\text{area}(S_\varepsilon)}\iint_{S_\varepsilon} u dS \leq M = u(x_0)$$

Since M is the maximum, it follows that u must be constant on the circle S_ε. This holds for all ε that stays within D, hence we get that u is a constant in the ball $B(x_0, r)$, where $r = d(x_0, \partial D)$. Next, choose a different point in $B(x_0, r)$ and repeat the argument to extend the region where u is constant. We repeat this process until we cover all of D.

(9) Repeating the steps in the Dirichlet case, we get $\nabla u = 0$, hence u is a constant. We cannot conclude this is zero since in the Dirichlet case, we knew u was 0 on the boundary but in the Neumann case, we have no such information.

(10) Let u be harmonic and w be any function such that $\frac{\partial w}{\partial n} = h$ on ∂D. Let $v = u - w$, note that $\frac{\partial v}{\partial n} = 0$. Then

$$\begin{aligned}
E[w] &= \frac{1}{2}\iiint_D |\nabla(u-v)|^2 dV - \iint_{\partial D} h(u-v)dS \\
&= \frac{1}{2}\iiint_D |\nabla u|^2 dV - \iint_{\partial D} hudS - \iiint_D \nabla u \cdot \nabla v dV + \iiint_D |\nabla v|^2 dV + \iint_{\partial D} hvdS \\
&= E[u] - \left(\iiint_D \nabla u \cdot \nabla v dV - \iint_{\partial D} \frac{\partial u}{\partial n} v dS\right) + \iiint_D |\nabla v|^2 dV \\
&= E[u] + \iiint_D v\Delta u dV + \iiint_D |\nabla v|^2 dV \\
&\geq E[u]
\end{aligned}$$